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TRAUMATIC ACID AND THIAMIN AS GROWTH FACTORS FOR ALGAE[1]

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Bottomley and, especially, Ashby[2] showed that Lemna develops well in a purely inorganic medium, although it thrives better if some organic matter, such as a soil extract, is added. The addition of soil extract and other chemically ill-defined organic matters[3] to the culture medium of algae is a general practice.[4] Since the chemistry of many plant hormones has become known in recent years it seemed of interest to see in which respect they affect the growth of algae. Yin[5] studied the effect of auxin on Chlorella and found an inhibition of the multiplication but an increase of the individual cell size. Pratt,[6] however, found just the reverse, an increase in cell number and no increase in cell size. Recently English, Bonner and Haagen-Smit[7] isolated and synthesized traumatic acid (1-decene-1, 10 dicarboxylic acid), a wound hormone of plants, which causes considerable proliferation and elongation of parenchymatous tissues (bean pods and potato tubers). On the assumption that this substance might exhibit similar properties on plant cells in general, it was added to the sterile cultures of unicellular algae[8] in order to determine whether it promotes the growth. Traumatic acid indeed proved to be highly effective in promoting multiplication of the culture, as is shown in the table for *Scenedesmus obliquus*, *Sc. bijugatus*, *Sc. brasiliensis* and *Palmellococcus miniatus*. Similar results were obtained with *Coccomyxa subellipsoidea* forma *putridena*, *Kirchneriella spec.?*, *Coelastrum proboscideum* var. *gracilis* and *Stichococcus bacillaris*.[9]

Thiamin was recognized as a growth factor for fungi by Schopfer[10] and later on also as a growth factor for higher plants.[11] Thiamin was found to be ineffective, or at least far less effective than traumatic acid, when added to cultures (in basic inorganic medium) of the previously mentioned algae. It proved, however, to be decidedly effective in promoting the growth of *Sphaerella lacustris* (*Haematococcus pluvialis*).[12] Traumatic acid also af-

fects the growth of *Sphaerella lacustris* but to a lesser extent than vitamin B_1.

Adenine has lately been shown to be active as a factor for leaf growth.[13] No algae have been found so far that respond markedly to it.

The above-mentioned experiments show that both traumatic acid and thiamin are growth factors for algae.

MULTIPLICATION OF ALGAE UNDER STERILE CULTURE CONDITIONS, AS MEASURED BY PHOTOELECTRIC TURBIDITY MEASUREMENTS

	ADDED TO THE BASIC INORGANIC MEDIUM:[8]							
	TRAUMATIC ACID			VITAMIN B_1		ADENINE	TRAUM. + B_1 + ADEN.	NO
	5000 γ/1	1000 γ/1	100 γ/1	1000 γ/1	100 γ/1	1000 γ/1	100γ/1 EACH	ADDITIONS
Scenedesmus obliquus	62.0	41.4	26.2	..	..	..	..	19.4
(6 days after inoculation; pH 6.6)	68.9	45.1	26.9	..	..	..	..	22.2
Sc. bijugatus	99.2	84.8	56.2	..	..	..	..	49.2
(12 days after inoculation; pH 6.6)	93.0	80.6	52.2	..	..	..	..	47.6
Sc. bijugatus (20 days after inoculation)	...	94.4	40.6	24.0	37.3	31.1	43.2	22.6
Sc. brasiliensis	141.4	100.0	71.4	..	..	..	..	74.2
(12 days after inoculation; pH 6.6)	118.0	92.2	70.2	..	..	..	..	62.4
Palmellococcus miniatus	115.6	107.0	49.8	..	..	..	..	56.4
(12 days after inoculation; pH 6.6)	102.0	92.6	48.0	..	..	..	..	43.2
Haematococcus pluvialis (ring on wall of tube estimated after 20 days)	...	Fourth	No growth	Best	Second	No growth	Third	No growth

In a purely inorganic medium and under the conditions described here the algae probably produce[14] a limited amount of the growth factors, but not sufficient to sustain a rapid growth.

[1] Work assisted by the Works Progress Administration (Official Project 65-1-07-98, Work Project N-11534).

[2] *Ann. Bot.*, **43**, 805–816 (1929).

[3] E.g., *Am. Jour. Bot.*, **27**, 161 (1940).

[4] Bold, Harold C., *Jour. Tenn. Acad. Sci.*, **12**, 205–212 (1936).

[5] *Proc. Nat. Acad. Sci.*, **23**, 174–176 (1937).

[6] *Am. Jour. Bot.*, **25**, 498–501 (1938).

[7] *Science*, **90**, 329 (1939); the traumatic acid used in these experiments was obtained through courtesy of these authors.

[8] The basic inorganic medium consisted of $^1/_5$ Hoagland solution plus minor elements, with the exception of copper (*Univ. Cal., Coll. Agri. Cir.* **347**, 36–37, December (1938)). Glass-distilled water was used throughout. Agar was not used in the experiments described here because it was found to be an unsuitable medium for the demonstration of the action of organic growth factors. Inoculations were made by using 1 cc. of a cell

suspension. The cultures were in culture tubes (15 cc. of medium) in indirect daylight in an air-conditioned greenhouse (constant temperature, 26°C.).

[9] These algae were obtained through courtesy of Dr. C. B. van Niel, Pacific Grove, Calif.

[10] Schopfer, W. H., *Compt. Rend. Soc. Phys. Hist. Nat. Geneve*, **51**, 26 (1934).

[11] Kögl, F., and Haagen-Smit, A. J., *Zeit. Physiol. Chem.*, **243**, 209–226 (1936); Bonner, J., *Science*, **85**, 183–184 (1937); Robbins, W., and Bartley, M., *Ibid.*, **85**, 246 (1937).

[12] Obtained through courtesy of Dr. Florence Meier Chase, Smithsonian Institution.

[13] Bonner, D. M., and Haagen-Smit, A. J., *Proc. Nat. Acad. Sci.*, **25**, 184–188 (1939).

[14] The presence of vitamin B_1 in marine algae was reported by E. R. Norris, M. K. Simeon and H. B. Williams, *Jour. Nutrition*, **13**, 425–433 (1937).

SOME TRABECULATE CODIUMS (INCLUDING TWO NEW SPECIES)

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Trabeculae ("Balken," "Cordons cellulosiques," etc.), as applying to protrusions from the "wall" into the cavity of a septate or unseptate coenocyte, occur particularly in species of *Caulerpa* and *Dictyosphaeria*, both genera of the general group of the Siphonales, to which the genus *Codium* also belongs. Thus far they have not been described for any of the numerous species of *Codium*. In a few species of *Codium*, however, various types of projections have been noticed extending from the inner portions of the apical membrane of the utricles into the general cavity of the coenocyte. O. C. Schmidt ("Biblioth. Bot.," Heft 91, p. 27, figure 7 (1923)) shows tips of utricles of his extended and probably composite *Codium adhaerens* which have the inner portions of the membrane alveolate, with inwardly extending anastomosing plates (as shown in section in the lowermost tip under his figure 7). Such sculpturing of the inner layers of the apical membrane has been seen by me in a number of species of both *Tylecodiums* and *Schizocodiums*, often sufficiently constant to seem characteristic of the particular species. M. A. Howe (in Britton and Millspaugh, "The Bahama Flora," p. 617 (1920)) has described a variety, or form, *cribosum*, of *Codium intertextum* Collins and Hervey, but this alveolation seems to be really a character of the mature utricles of this species (which is included under the extensive aggregate of *Codium adhaerens* by O. C. Schmidt). These alveolar inner surfaces of the apical membrane have been little noticed and often are not to be seen except on careful illumination. From the surface, under proper lighting of fresh utricles or those swollen to nor-

mal size and shape, they look like the sieve plates of the higher plants. It seems best to regard them as lamellate anastomosing projections, rather than "pitting," and to consider them under the head of complex trabeculae.

Another type of trabeculate growth is shown in the peculiar local thickening of the inner apical membrane of the *Codium Schmidtii* Vouck (V. Vouck, *Acta Bot., Inst. Bot. Univ. Zagrabensis*, **10,** 9–12, plate 1, text figures 1–4 (1935)), who states, accurately so far as I know, that "such a peculiarity of utricle structure has never been recorded." The apical membrane of the utricle thickens, but at the center the thickening projects into the cavity of the utricle in the form of a boss, or button. Unfortunately for this proposed new species of Vouck, I find that the utricles of the type specimen of *Codium Muelleri* of Kuetzing show the same thickened structure, although neither Kuetzing (*Tab. Phyc.*, **6,** 34, plate 95, II (1856)) nor O. C. Schmidt (loc. cit., 51, 52) figures it. Kuetzing has nothing in his description to suggest that, "keen-eyed" as he undoubtedly was, he had noticed anything of this inwardly projecting blunt umbo, but Schmidt (loc. cit., p. 51) evidently saw it and failed to interpret it properly, although he quotes Areschoug's statement (*Act. Reg. Soc. Scient.*, ser. 3 (1), 368, Upsala (1854) sub *Codium tomentosum australasiacum* Aresch.) concerning an apical "mamilla" on the utricle separated by a transverse septum from the utricle itself. O. C. Schmidt seems to regard this "mamilla" to be due to an illusive appearance in the collapsed broad utricles of the species, not properly swollen out. Vouck's interpretation of the structure agrees with my own, but his binomial *C. Schmidtii* must, therefore, be relegated to the synonymy of *C. Muelleri*. It may be of interest to notice that *C. Schmidtii* Vouck is a topotype of *C. Muelleri* Kuetz., since both were collected on shores of the Lefever (or Lefebre) Peninsula by Ferdinand Mueller, the former in "July, 1882," the latter in "July, 1852."

When one reads the descriptions of the utricles of *Codium globosum* of A. H. S. Lucas (*Proc. Linn. Soc. of New South Wales*, **52,** (pt. 4), 558, plate 12, figure 4 (1927)) one will not find any mention of the scattered, short, broadly conical trabeculae projecting from the inner surface of the apical

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EXPLANATION OF PLATE

Codium Cranwelliae Setchell

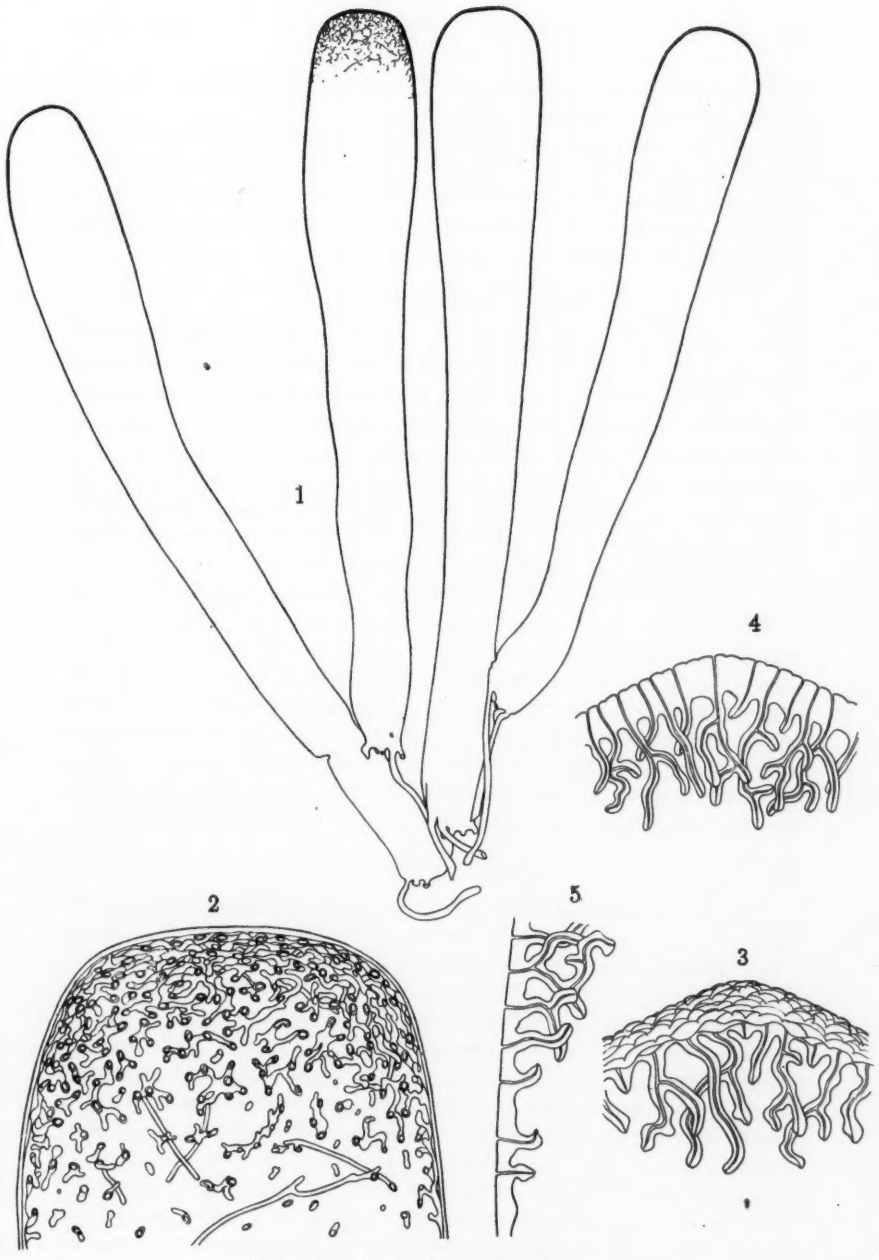
1. Group of utricles to show size, shape and branching. Trabeculae are represented in only one while the rest are left blank. × 25 diam.

2. Uppermost portion of utricle with trabeculae from the surface and, in part, in optical section. × 125 diam.

3. Thick section of extreme tip of utricle, showing outer surface and trabeculae. × 250 diam.

4. Thin section of extreme tip of utricle. × 250 diam.

5. Thin section of lower part of trabeculate zone of side of upper part of utricle. × 250 diam.



(See explanation of plate on opposite page)

membrane. A careful examination of the utricles of specimens of the type collection, however, does show them.

While the "trabeculae" of the three species thus far listed may seem too insignificant to warrant applying this term to them, the structure seems more definitely worthy of the name as applied in the case of the two species now to be described, viz., *Codium phasmaticum* from Hawaii and *C. Cranwelliae* from New Zealand.

Codium phasmaticum sp. nov.

C. thallo applanato, difformi, inferne arcte adhaerente, margine lente undulato, spongioso, 3–5 cm. diam., circum 0.5 cm. crasso, laetevirente; utriculis laxe corymboso-ramosis, dimorphis, (a), *macro-stenophysis, cylindricis, 62.5–93.75 μ latis et 500 μ usque ad 1 mm. altis,* (b), *macro-euryphysis, obovato-pyriformibus, superne globoso-intumescentibus, 125–250 μ latis, 500 μ usque ad 625 μ altis; membrana apicali usque ad 5 μ crassa, intus 1–2 usque ad 7–8 trabeculis cylindricis, simplicibus, gracilibusque usque ad 15 μ longis et 2.5–3 μ crassis adornatis; sub apice 62.5–65 μ, pilis in singulo usque ad 3–4 verticillis, circumdatis; gametangiis adhuc non visis.*

In saxis littoralibus, apud "Laupahoehoe," in insula "Hawaii," a W. A. Setchell lecto, No. 10034 (spec. typicum Hb. Univ. Calif. 468720).

Only a single specimen was found on rocks which were almost bare of any Algae. Each utricle, however, shows the relatively long and slender, although sparse, inwardly projecting processes whose structure is that of the simpler trabeculae of *Caulerpa*, or even more like those of certain species of *Dictyosphaeria* (cf. Weber van Bosse, "Nuova Notarisia," ser. 16, 144, Oct. (1905), Siboga Exp., Mon. LIX *a*, 64 (1913); Boergesen, "Mar. Alg. Danish W. I.," **1**, 39 (1913); Setchell, "Univ. Calif. Pub. Bot.," **12**, 79 (1926)).

Codium Cranwelliae sp. nov.

"A new Codium," Cranwell and Moore, *Trans. Roy. Soc. N. Z.*, **67**, 399 (1938).

C. thallo compresse-ovoideo-globoso, 0.5–5 cm. longo, 0.5–3 cm. lato, 0.5–2.5 cm. alto, erecto, superne libero, per basim moderate latam affixo, nitente laetevirente et spongioso; filamentis internis 35–40 μ latis, dense intertextis; utriculis superficiem liberam totam investientibus, parce ramosis, elongato clavatis, megistophysis, 5000 μ usque ad 6500 μ longis, 625 μ latis, membrana apicali, et subapicali leviter incrassatis et trabeculas numerosas, simplices ramosasve emittente; pilis deuntibus; gametangiis nondum visis. (Cf. plate, p. 445.)

In rupis concavis, zonis medio-littoralibus, in insulis "Poor Knights," N. Z., No. 205, 206, 207, Cranwell et Moore, et ab undis rejectis, "Long Beach, Bay of Islands," N. Z., No. 182, 791, 1557, V. W. Lindauer. (Spec. typicum in Hb. Univ. Calif., No. 564498).

Codium Cranwelliae is a truly remarkable species. It has the habit and the huge ("megistophyse") utricle associated with the *Bursa* group within the genus. Its utricles are distinctly branched although slightly, and this is true even of *Codium Bursa* itself although O. C. Schmidt ("Bibliotheca Botanica," Heft 91, pp. 23, 24, 25 (1923)) in his monographic study of *Codium* distinguishes the "*Adhaerentia*" as having the "Blasen reich verzweight" and the "*Bursae*" as well as the other two sections as having "Blasen stets unverzweight." I find, however, that *Codium Bursa* (L.) Ag. and *C. mammillosum* Harv. have utricles distinctly although only slightly branched and many species included in Schmidt's "*Adhaerentia*" other than *C. adhaerens* in Schmidt's extended sense, can scarcely be said to have the utricles really "richly" branched. In habit, *Codium Cranwelliae* is truly of the *Bursa* group, being cushion-shaped and with a limited basal area of attachment, even somewhat more so than *C. spongiosum* Harv. of Australian waters, which it closely resembles, except that it seems more densely packed with filaments in its interior, has a narrower attachment, and has the conspicuously trabeculate utricles. From *Codium Perrinae* A. H. S. Lucas (*Proc. Linn. Soc. N. S. W.*, **60,** 203, figure 4 (1935)), apparently closely related to *C. spongiosum*, it differs in lack of crescentic habit and the longer and trabeculate utricles not thickened at the apex.

The utricles in *Codium Cranwelliae* are very long (up to 6 or 7 mm.) and are "megistophyse" in one sense but they are not as broad as those of the truly megistophyse species of the *C. mammillosum* group of Australia, Japan and South Africa, which approach a maximum of 2–3 mm. in diameter. It is, however, the occurrence of undoubted trabeculae, not only crowded but with one or more branches, which sets off *C. Cranwelliae* from all other Codiums thus far described. This has led me to review the various Codiums which have been accumulating for some years in my studies of this genus and which had led me to survey all the species known to me to have "projections" inwardly from the apical or subapical membranes and consider them under the head of "trabeculate" Codiums and to associate these structures with the various inwardly projecting cylindrical protrusions of the inner cell walls in the various members of the Siphonales. The structure of the trabeculae in *Codium* seems to be identical with those in *Caulerpa*, in that a distinct core (or tube?) is present. The trabeculae are always on the apex of the cavity of the utricle, protruding from the inner surface of the apical or subapical membranes, limited in growth, simple or branched, and ending free. Oltmanns ("Morph. u. Biol. d. Algen," I (Ed. 2), 413–415 (1922)) summarized the literature and opinions as to structure and function of the trabeculae in *Caulerpa*, up to his time of writing. As to what their function may be in the utricles (and only in the utricles) of *Codium*, the writer has as yet no suggestions, especially since he has seen only dead plants. The fact of their occurrence and particularly their luxuri-

ant development in *Codium Cranwelliae* may add information to be considered in any attempt to solve the problem.

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*STUDIES IN MINERAL METABOLISM WITH THE AID OF INDUCED RADIOACTIVE ISOTOPES. IV—MANGANESE**

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This communication is a report of a test study of the suitability of radioactive manganese for biological tracer investigations. The requirement for manganese of the animal organism is very small. Consequently, a high specific radioactivity is required in the samples in order to keep the test dosages within physiological limits.

Various investigators have demonstrated that manganese is essential for the health and well-being of the animal organism.[1, 2, 3, 4] However, little is known about its specific biological functions. Orent and McCollum[5] found that manganese aids lactation and prevents degeneration and atrophy of the testes in the rat. In the chick, it is markedly effective in preventing the development of the bone condition known as perosis.[6] Manganese has been found to serve as an activator of certain enzymes, notably arginase,[7, 8] phosphoglucomutase,[9] and certain peptidases.[10]

The results reported here demonstrate that radioactive manganese may be usefully employed in the elucidation of many problems connected with its metabolism.

Experimental Methods.—The isotope Mn^{54}, with a half life of 310 days,[11] was employed in this investigation. It was prepared in the Radiation Laboratory of the University of California by bombardment of iron with deuterons. The reaction involved is:

$$_{26}Fe^{56} + {}_{1}H^{2} \longrightarrow {}_{25}Mn^{54} + {}_{2}H^{4}$$

Radioactive manganese was isolated from the residue after extraction of the radioactive iron. Traces of radioactive phosphorus formed from iron phosphide were removed by dissolving the iron-free residues in $0.5M$ HNO_3, adding small quantities of NaH_2PO_4 to serve as a carrier, and precipitating twice with a solution of $Bi(NO_3)_3$ in $0.5M$ HNO_3. The filtrate was evaporated to dryness, dissolved in $16M$ HNO_3, and the manganese precipitated as MnO_2 with a few crystals of $KClO_3$. The MnO_2 was filtered off on a Jena sintered glass filter, then dissolved with H_2O_2, and reprecipitated twice by

the above procedure to free it from traces of radioactive cobalt that were also present in the original residues.

Finally the MnO_2 was dissolved in dilute HCl, evaporated to dryness to remove excess acid and then dissolved in enough water to give a concentration of about 1 mg. of manganese per ml. of solution. The radioactivity of the resulting preparation was of the order of 0.1 μc. per mg. of Mn. All radioactivity measurements were made with a thin copper wall Geiger-Müller counter and a scale of eight circuit.

TABLE 1

EXCRETION OF LABELED MANGANESE

TIME, HOURS	ORAL ADMINISTRATION, PER CENT OF TOTAL DOSE	INTRAPERITONEAL INJECTION, PER CENT OF TOTAL DOSE
	Feces	Feces
0–23	4.2 ± 0.26	*
23–48	50.2 ± 0.45	53.9 ± 0.60
48–75.5	39.9 ± 0.40	36.8 ± 0.35
Total excreta	97.2† ± 0.68	90.7 ± 0.70

* Amount found was not statistically significant.

† Total excreta includes 2.9 per cent labeled manganese found in the urine in the time interval between 11 and 23 hours. This was the only statistically significant amount found in any of the urine samples.

For the metabolic tests, 1-ml. doses of the labeled manganese chloride (1 mg. Mn) were administered to each of two rats weighing about 160 gm., one by stomach tube, the other by intraperitoneal injection. During the experiment, the animals were maintained on a synthetic control diet.

The feces and urine were obtained separately at the desired intervals through use of glass separators.[12] At the end of 75.5 hours, the animals were anesthetized with ether, blood was drawn by cardiac puncture and the animals were sacrificed. The various organs were dissected out, the skin was removed and the muscles and bone were separated by boiling the residual carcass in 1:5 NH_4OH, and then allowing the solution to stand for several days.

All tissues were dried and dry ashed at 500°C. One mg. of inactive manganese as the sulfate was added to each ashed sample as a carrier for the radioactive Mn. The ash was dissolved in a minimum of dilute HCl and filtered. The total filtrate was evaporated to dryness with concentrated HNO_3 three times to remove the chloride. The residue was then dissolved in hot concentrated HNO_3, and a few crystals of $KClO_3$ were added to precipitate the MnO_2. The precipitate was filtered off, another mg. of inactive Mn was added to the filtrate and MnO_2 was again precipitated. This second precipitate was collected on the same filter paper so that all of the radioactivity in a single tissue was on the one filter paper.

In the case of the bones, it was necessary first to separate the manganese from the large amounts of calcium present. The bone ash was dissolved in dilute HCl and evaporated to dryness. The chlorides were dissolved in water in an Erlenmeyer flask and NH_4Cl was added to the neutral solution. A freshly prepared $(NH_4)_2S$ solution was added to precipitate the manganese sulfide. The flask was filled with boiled water and stoppered. After standing for 12 hours, the precipitate was filtered off. The MnS was dissolved in a small amount of dilute HCl, and then the MnO_2 was separated by the same procedure as in the case of the other tissues. An aliquot of the original radioactive $MnCl_2$ solution was treated in the same manner as the test samples to serve as a standard for comparison of radioactivity measurements.

The total radioactive manganese recovered from each animal was computed from the measured activities of the excreta and of all the tissues. The recovery was 96 per cent in the case of oral administration, and 78 per cent in the case of intraperitoneal injection, so that the data have been corrected by the factor 100/96 and 100/78, respectively, to make them comparable.

Results.—The course of excretion of the labeled manganese over the 75.5 hours is shown in table 1. Most of the manganese is excreted in the feces whether it is administered orally or by injection. This agrees with Skinner, Peterson and Steenbock,[13] who have reported that from 80 to 99 per cent of orally fed manganese was excreted in the feces, depending upon the amount ingested. Except for statistically insignificant traces, labeled manganese was found in the urine only in the second collection period of the animal given the dose orally.

From the data it is not possible to decide to what extent manganese is absorbed from the gastro-intestinal tract when it is administered orally, although it is probable that it is small. When administered intraperitoneally, 90.7 per cent of the labeled manganese appeared in the feces, showing that there is a preferential excretion into the alimentary tract.

Table 2 shows the distribution of the labeled manganese that was retained by the animal. When administration was oral, the 2.8 per cent retained manganese was found in the liver, bone, muscle and blood, the liver showing the largest uptake.

When administration was by injection, the retained manganese was found in the skin, bone, liver, muscle, small intestine and stomach, the skin and bone showing rather large amounts. Other tissues showed no significant amount. In general, these observations also agree with the findings of Skinner, Peterson and Steenbock.[13]

The manganese found in the stomach and small intestine possibly represents manganese in the process of being excreted. Muscle and skin, apparently, are important sites for the storage of manganese that is absorbed,

especially as these tissues represent a large portion of the mass of the animal. Bone and liver also seem to be important in the storage of manganese. The manganese found in the liver may be indicative of its excretion into the bile, or it may be connected in some manner with the activation of certain enzymes found in the liver.

Summary.—1. Radioactive manganese, Mn^{54}, is suitable for "tracer" studies on the metabolism of manganese.

2. On a normal control diet the rat excreted over 90 per cent of the manganese within 75 hours, when administered either by stomach tube or by intraperitoneal injection.

3. Very little, if any, of the absorbed manganese is excreted in the urine.

4. Liver, bone and muscle take up appreciable quantities of the absorbed manganese. Other tissues may take up varying amounts of the manganese, due to storage or to processes of excretion.

TABLE 2

DISTRIBUTION OF LABELED MANGANESE
(In Per Cent of Total Dose, 75.5 Hours after Administration)

	ORAL ADMINISTRATION			INTRAPERITONEAL INJECTION		
TISSUES	WEIGHT, GM.	CONTENTS IN WHOLE TISSUE	CONTENTS PER GM., FRESH WEIGHT	WEIGHT, GM.	CONTENTS IN WHOLE TISSUE	CONTENTS PER GM., FRESH WEIGHT
Muscle	95	0.7 ± 0.12	0.007 ± 0.0012	97	0.8 ± 0.16	0.008 ± 0.0016
Bone	8.6	0.7 ± 0.11	0.081 ± 0.013	9.8	2.0 ± 0.17	0.20 ± 0.017
Skin	30.66	*	*	28.94	3.7 ± 0.22	0.13 ± 0.008
Whole blood	5.14	0.5 ± 0.14	0.097 ± 0.027	5.07	*	*
Heart	0.50	*	*	0.56	*	*
Liver	6.49	0.9 ± 0.15	0.14 ± 0.023	8.54	1.2 ± 0.20	0.14 ± 0.023
Small intestine	5.37	*	*	6.36	0.8 ± 0.27	0.13 ± 0.042
Large intestine	1.13	*	*	1.23	*	*
Stomach	4.75	*	*	3.67	0.8 ± 0.27	0.22 ± 0.074
Spleen	0.35	*	*	0.89	*	*
Kidney	1.38	*	*	1.36	*	*
Lung	0.88	*	*	1.16	*	*

* Radioactivity measurements made, but amounts found were not statistically significant.

We are indebted to Professor E. O. Lawrence and Dr. M. D. Kamen of the Radiation Laboratory of the University of California for the radioactive manganese used in this investigation.

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A REVERSION TO WILD-TYPE ASSOCIATED WITH CROSSING-OVER IN DROSOPHILA MELANOGASTER

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In a study of two alleles of lozenge eye in *Drosophila melanogaster*, a low frequency of reversion to the wild-type has been observed. The reversion involves the color and structure of the eye, and also the genital tracts which are abnormal in the mutant females.[1] In each case in which the reversion has occurred, crossing-over between the *X*-chromosomes has also occurred.

Glossy and spectacle are sex-linked, recessive mutants, alleles of lozenge, which were induced by irradiation. Glossy (lz^g) individuals have eyes which are blood-red in color, with fused facets making a glossy surface. Spectacle (lz^s), as reported by Dr. J. T. Patterson,[2] is characterized by a light brown color of the eye, and the facets are run together to cause a smooth surface of the eye. In the compound, heterozygous glossy-spectacle females, glossy is more dominant in its expression; and the character of the eye is nearly that of homozygous glossy. Spectacle is associated with the *dl*-49 inversion, and the mutant gene is located within that inversion. Glossy is also located within an inversion which is very similar to, probably the same as, the *dl*-49 inversion.

From the mating of lz^g *Bx*/lz^s *f* females to lz^g *Bx* males, a total of 5584 offspring have been inspected. Most of the offspring were of the expected types. Males were glossy or spectacle. Females were phenotypically

alike, although genotypically they were either homozygous glossy or heterozygous glossy-spectacle. Eleven of the 5584 offspring, two males and nine females, were wild-type for the mutant eye and genital traits. The wild-type traits persist whether the individuals are heterozygous for spectacle or glossy. Two of the exceptional flies were tested for crossing-over with a combination of genes in a non-inverted X-chromosome. The results indicated that the inversion had not been lost.

The appearance of each exceptional, wild-type offspring was associated with crossing-over. Ten of the exceptional offspring were Beadex (*Bx*), and carried, therefore, the right end of the glossy-bearing X-chromosome. The other one, forked (*f*), probably was the result of a double crossover. In a test-cross using yellow and Hairy wing as markers for the left end of the chromosome (*y Hw* lz^s *f*/lz^s *Bx*), two out of 305 offspring were of the reversed type. One, a male, was *y Hw Bx;* and the other, a female, proved to be heterozygous for that combination. The flies showing the reversion to wild-type have an X-chromosome which is composed of the right end of the glossy-bearing chromosome and the left end of the spectacle-bearing chromosome. Moreover, the associated crossing-over occurs within the inverted regions. The loci for miniature (*m*) and vermilion (*v*) are located in that order within the inverted regions. From the mating of *v* lz^s *f*/*m* lz^s *Bx* females to glossy males, two exceptional males occurred among 1285 offspring, and both were *v Bx*. Apparently the crossover occurs between *v* and one of the alleles of lozenge.

Although crossing-over seems to be associated with the reversion, and one crossover type appears, the complementary type has not been recovered. It is not known whether the failure to recover the complementary type is due to the inability of that type to live, or to the inability of the observer to recognize the combination. Glossy is almost completely dominant over spectacle. The presence of glossy and spectacle in one chromosome, with either glossy or spectacle in the other chromosome, may produce a phenotype so similar to homozygous glossy and the compound glossy-spectacle that it will be difficult, if not impossible, to differentiate the three genotypes.

It does not seem possible to explain the reversions on the basis of mutations in the genic sense. The exceptional type has not appeared among the offspring of either homozygous glossy or spectacle; nor has it been observed among the offspring from the females heterozygous for either glossy or spectacle and lozenge. Lozenge is present in a chromosome which has no inversion. The only mating which has given the exceptional, wild-type offspring has been the compound glossy-spectacle. Each of the mutants is present in an inversion, and the inversions are at least almost identical. Under such conditions, crossing-over is expected to occur throughout the chromosome. The crossing-over associated with the reversion occurs

within the inverted region, and as shown by the appearance of the *v Bx* crossover type involves the region near the alleles glossy and spectacle.

Although crossing-over is an active factor in the reversion of the alleles to the wild-type, it is not possible as yet to determine the exact nature of the phenomenon. The condition can be a case of unequal crossing-over;[3] but it can as likely be a case which involves the "repeat" hypothesis developed by Bridges,[4] in which different primary loci of the chromosome are involved in the expression of the two mutants.

Summary.—1. A reversion to wild-type occurs with a frequency of 0.2% among offspring from a female which had glossy on one and spectacle on the other *X*-chromosome. Both traits are recessive and are allelic to lozenge.

2. The reversion is always associated with crossing-over.

3. The associated crossing-over occurs within the inverted regions of the chromosome, and at or near the lozenge locus.

4. The suggestion is made that the results probably involve either unequal crossing-over or crossing-over between "repeats."

[1] Oliver and Green, *Anat. Rec.*, **75** (Supp.), 100–101 (1939).
[2] Patterson and Muller, *Genetics*, **15**, 495–578 (1930).
[3] Sturtevant, *Ibid.*, **10**, 117–147 (1925).
[4] Bridges, *Jour. Heredity*, **29**, 11–13 (1938).

HD 167362, AN OBJECT SIMILAR TO CAMPBELL'S HYDROGEN ENVELOPE STAR

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Campbell's famous hydrogen envelope star[1] consists of a Wolf-Rayet nucleus of type *WC*8, which is surrounded by a nebula showing strong forbidden [N II] lines; in fact, the two red [N II] lines contribute more than Hα to the red color of the envelope. This association seems rather peculiar, the exciting Wolf-Rayet nucleus showing no trace of nitrogen,[2] whereas the nebulosity around it is very rich in nitrogen.

We have found that spectroscopically HD 167362 (CD $-30°$ 15469; MWC 288; $m = 11.8$) is a similar object. Its distance is too great to resolve the nebula, and the widths of the lines belonging to the nucleus are not as much broadened as they are in Campbell's star. The bright lines were discovered by Keeler[3] at the Harvard Observatory and were first de-

scribed by Miss Fleming,[3] who called attention to the similarity with η Carinae. Miss Cannon[4] listed the spectrum among the P Cygni stars, and recorded the presence[5] of [O III] 5007 and 4959, and of other faint emission lines.[6] Miss Hoffleit[7] found no change in the apparent magnitude of the star. Merrill and co-workers[8] found that H_α is bright. C. Payne Gaposchkin and S. Gaposchkin[9] list the star among objects possessing forbidden [Fe II], but the source of their observation is not given. Our plates do not show [Fe II]. Apparently, no detailed study of this interesting spectrum has thus far been published.

Five spectrograms were obtained at the McDonald Observatory, from April 24 to May 2, 1940; three were obtained with the quartz prisms (dispersion 100 A/mm. at λ 3933) and two with the glass prisms (dispersion 50 A/mm. at λ 3933).

The spectrum consists of a weak continuum, with maximum intensity near λ 4300, on which are superimposed strong, bright lines. The forbidden lines and the lines of H seem to be a little sharper than the He I lines and He II 4686. Still broader features belong to C III and C IV. The measured lines are collected in table 1.

The object consists of a Wolf-Rayet nucleus surrounded by a nebula, and we distinguish three groups of lines, which we shall designate as *A*, *B* and *C*.

A. Lines Originating in the Nucleus Only.—This is the case of the broad lines of He II, C III, C IV and O III; the Wolf-Rayet nucleus is a typical carbon star; according to the intensity ratios adopted for classification purposes, the spectral type is probably later than the latest type *WC*8 in Beals' sequence. Since the band width is much smaller than 10 A and since He I plays certainly an important rôle in the nuclear part of the spectrum (see group *C*), we should be inclined[10] to assign the central star to type *WC*9. There is no trace of N II, N III, N IV or N V. Several lines of the nucleus have P Cygni character.

B. Lines Originating in the Nebula Only.—The forbidden lines of the following elements (in order of decreasing intensity) are observed in the nebula: [N II], [O II], [O III], [S III], [O I]. The transitions of the nebular type are much stronger than those of the auroral type (for example, in [O III] and [N II]). This is the usual thing in planetary nebulae. The transauroral transitions of [S II] may be present, but are blended with C III 4069 and O II 4076. The mean ionization from strong lines of [N II] and [O II] is around 14 volts; but the forbidden lines of neutral [O I] and doubly ionized [O III] and [S III] are fairly strong.

C. Lines Belonging to the Nucleus and to the Nebula.—The lines of He I and H are not exclusively of nebular origin. This conclusion is based upon the following facts: (1) The lines of [N II] and [O III] are sharper than those of He I, as may be seen from a comparison of He I 5876 and N II

TABLE 1

SPECTRUM OF HD 167362

STAR		IDENTIFICATION			
λ	Int.	Element	λ	Int.	Notes
3609.5	1*n*	C III	3609.61	5	
		C III	3608.96	4	
3682.7	1	H_{20}	3682.81		
3687.0	1	H_{19}	3686.83		
3691.3	1	H_{18}	3691.56		
3697.0	1	H_{17}	3697.15		(1)
3703.8	1–2	H_{16}	3703.85		(1)
3711.9	2	H_{15}	3711.97		(1)
3715.	0*n*	O III	3715.08	6	
3721.8	2	H_{14}	3721.94		(1)
3725.9	8	[O II]	3726.2		
3728.6	6	[O II]	3729.1		
3734.2	2	H_{13}	3734.37		
3750.4	2–3	H_{12}	3750.15		
3755.2	1	O III	3754.67	7	
3761.	1*n*	O III	3759.87	9	
3770.9	3	H_{11}	3770.63		
3791.9	1*n*	O III	3791.26	6	
3797.9	3–4	H_{10}	3797.90		
3819.4	2	He I	3819.61	4	
3835.4	4	H_9	3835.39		
3867.6	0	He I	3867.46	2	
3888.7	4	He I	3888.65	10	(4)
3889.1	4	H_8	3889.05		
3970.2	6	H_ϵ	3970.08		(1)
4026.3	2	He I	4026.19	5	
4057.0	1*n*	C III	4056.06	5	
4068.7	3*n*	C III	4067.87	9	(1)
		C III	4068.97	9–10	
		C III	4070.30	10	
		[S II]	4068.5		
4075.7	1–2	[S II]	4076.5		
		O II	4075.87	10	
4101.8	8	H_δ	4101.75		
4120.	1	He I	4120.81	3	
4144.	0	He I	4143.77	2	
4155.4	1	C III	4156.50	4	
		C III	4152.43	3	
4183.4	2*A*	C III	4187.05	10	(1)
4186.0	2*E*				
4340.7	10	H_γ	4340.48		(1)
4362.7	1	[O III]	4363.21		
4388.	1	He I	4387.93	3	
4468.7	3*A*	He I	4471.48	6	(1)
4471.3	3*E*				
4516.7	1	C III	4516.5	4	

4651.	4*nn*	C III	4647.44	20	(1, 2)
		C III	4650.26	19	
		C III	4651.48	18	
4658.8	2*n*	C IV	4658.64	5	
4667.	1*n*	C III	4665.90	6	
4686.	1*n*	He II	4685.81		
4861.3	10	H_β	4861.34		
4958.9	4*s*	[O III]	4958.91		
5006.9	7*s*	[O III]	5006.84		
5696.	3*n*	C III	5696.0	8	
5755.3	3*s*	[N II]	5755.0		
5802.	0*n*	C IV	5801.51		(3)
5876.0	5	He I	5875.62	10	
6300.	2	[O I]	6300.2		
6311.	3	[S III]	6310.2		
6548.	7	[N II]	6548.4		
6563.	15	H_α	6562.82		
6584.	10	[N II]	6583.9		
6678.	1	He I	6678.15	(6)	

Identifications in square brackets designate forbidden transitions.

1—Line showing the P Cygni character.

2—Broad line extending over 4.1 *A*.

3—For this identification see the discussion of the spectrum of NGC 6543 and its nucleus; P. Swings, *Ap. J.* (in press).

4—Blended.

5755. (2) Several lines of H and He I show P Cygni character, the velocity of ejection being practically the same as that given by the nuclear lines.

The Balmer lines are followed to H_{20} and a strong Balmer continuum extends to λ 3450.

The lines of the nucleus and of the nebula seem to have the same radial velocity. The mean value from 18 lines is −24 km./sec. The ejection velocities obtained from He I 4471 and C III 4187 are 176 and 186 km./sec., respectively. Since these values are practically equal we adopt for the ejection velocity 180 km./sec.

The great similarity of HD 167362 to Campbell's star (BD + 30° 3639 = HD 184738) is evident at once. In both cases, the nucleus is a late *WC* star and the nebula shows very strong [N II] and [O II] lines. In BD + 30° 3639, H_α is weaker than the total intensity of the two red [N II] lines, but this may not be true for HD 167362. But it should be remembered that in HD 167362, H_α belongs simultaneously to the nebula and to the nucleus; hence the relative contributions of H_α and [N II] in the nebula are probably not very different. In both stars the nucleus shows He I lines; the spectroscopic separation between nucleus and nebula is easier in the case of Campbell's star, where the two sources are clearly separated in a large telescope and where the widths of the lines are very different. In Campbell's star, He I is practically absent from the nebular part of the spectrum, while

it is present in the nucleus. [O III] is strong in HD 167362, and very weak in Campbell's star.

The striking association in HD 167362 and BD + 30° 3639 of a carbon nucleus with a nitrogen envelope suggests that a comparison with NGC 6543 would be interesting, despite the higher excitation prevailing in the nuclear and nebular parts of NGC 6543.[11] This object also shows strong nebular lines of [N II], but its nucleus exhibits both N IV and C IV with similar intensities.

[1] *Astronomy and Astrophysics*, **13**, 461 (1894).

[2] Several excellent spectrograms of Campbell's star (BD + 30° 3639) have recently been secured at the McDonald Observatory and agree closely with the description of the spectrum by Wright (*Lick Obs. Pub.*, **13**, 220 (1918)); there is no trace of N II, N III, N IV or N V in the nucleus, which is a typical carbon star.

[3] *Ap. Jour.*, **2**, 354 (1895).

[4] *Harvard Ann.*, **76**, 31 (1916).

[5] *Harvard Circ.*, No. 224 (1921).

[6] Henry Draper Catalogue.

[7] *Harvard Bull.*, No. 892, 20 (1933).

[8] *Ap. Jour.*, **61**, 389 (1925); **76**, 156 (1932).

[9] "Variable Stars," *Harvard Obs. Monograph*, No. 5, 311 (1938).

[10] Beals has chosen the numbering from *WC*6 to *WC*8 so as to allow a certain latitude for new discoveries at either end of the sequence. See *Trans. I. A. U.*, **6**, 248 (1938).

[11] P. Swings, *Ap. Jour.* (in press).

THE SPECTRUM OF RW HYDRAE

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RW Hydrae[1] is an abnormal long-period variable having an unusually small range, of about one magnitude; the maximum photographic magnitude is 9.7 to 9.9 and the minimum 10.8 to 10.9. It has a late-type spectrum upon which are superimposed several bright lines. Miss Cannon's estimates of the spectral type range from *K*5 to *M*2; she noticed bright H_β, H_γ and H_δ.

The spectrum has been investigated by Merrill,[2] who observed, besides the late-type spectrum, several bright lines of H, He I, He II and [O III] (auroral transition λ 4363 only).

This object offers a striking similarity to AX Persei, CI Cygni, Z Andromedae, R Aquarii, T Coronae and others, consisting of a late-type star and of a companion of high excitation. In previous papers[3] we have given a

new discussion of several binaries of this type, using material secured with the 82-inch reflector of the McDonald Observatory. The present note brings similar information concerning RW Hydrae, based on four spectrograms obtained between April 19 and 25, 1940; two were taken with the quartz prisms (dispersion 100 A/mm. at λ 3933) and two with the glass prisms (dispersion 50 A/mm. at λ 3933).

During the interval covered by our observations, the late-type component was of spectral class *M*0 or late *K;* compared with the bright lines, the late-type spectrum was much stronger than in AX Persei or CI Cygni, which we observed between September, 1939, and February, 1940. The red component completely obliterates the region above λ 4500, so that only strong bright lines may be detected above that wave-length.

The emission lines are collected in table 1. Besides the features observed by Merrill, our spectrograms reveal many permitted O III-transitions, weak [O II], fairly strong [Ne III], weak N_1 (nebular transition of [O III]), weak Si I and Ca II. All the emission lines are sharp.

The Balmer series is clearly seen in emission to H_{22}, and a strong Balmer continuum extends to λ 3300. Many He I lines are present. The Si I line, λ 3905, which also appears in AX Persei and CI Cygni, belongs presumably to the red-variable component; it seems rather probable that the weak Ca II lines are also excited in the atmosphere of the red star.

Besides the strong auroral transition of [O III], we observe a very weak nebular transition, N_1. Thus, the relative intensities of λ 4363 and N_1 are of the type found in planetaries of class Pd, such as IC 4997. Similar relative intensities were observed in AX Persei, CI Cygni, Z Andromedae and R Aquarii, which belong to the same group of binaries.

Fourteen permitted lines of O III are identified between λ 3265 and λ 3962. This spectrum is not excited by Bowen's fluorescence mechanism to any appreciable extent, because we find the lines of the singlet, triplet and quintet systems with the normal intensities of a recombination spectrum. The ionization potential of O^{++} is 54.6 volts, which is close to that of He^+, namely, 54.1 volts. The line He II 4686 is strong. So far as we know, no planetary nebula has been found showing such a complete recombination spectrum of O III. For example, in the extensive investigation by Bowen and Wyse[4] the only recombination line of O III which they observed is λ 5592 ($3s^1P^o - 3p^1P$); because of the blending with the red component, we are unable to discuss the region of λ 5592, but other recombination lines, such as λλ 3774, 3791 and 3962 are not found in the table by Bowen and Wyse.

It is possible that the permitted O III lines do not belong to a nebula, but rather to the exciting nucleus. Both the O III and the N III lines, and perhaps also those of He II and partly those of H and He I may belong to a nucleus of type *WN* possessing abnormally sharp lines; such a *WN* nucleus

TABLE 1

BRIGHT LINES IN RW HYDRAE

STAR			IDENTIFICATION	
λ	Int.	Element	λ	Int.
3266.	1	O III	3265.45	10
3312.4	0	O III	3312.30	5
3341.1	1	O III	3340.74	6
3383.0	1	O III	3382.69	3
		O III	3383.85	2
		O III	3384.95	4
3429.3	1–2	O III	3428.67	3
		O III	3430.60	4
3444.0	3	O III	3444.10	5
3676.01	1	H_{22}	3676.36	
3679.41	2	H_{21}	3679.35	
3682.55	2	H_{20}	3682.81	
3686.57	2	H_{19}	3686.83	
3691.38	2	H_{18}	3691.56	
3696.91	2	H_{17}	3697.15	
3703.77	2–3*n*	H_{16}	3703.85	
		O III	3702.75	5
		O III	3703.37	5
3707.13	0–1	O III	3707.24	6
3711.80	3	H_{15}	3711.97	
3714.60	0	O III	3715.08	6
3721.75	3	H_{14}	3721.94	
3724.90	1	[O II]	3726.1	
3734.36	4	H_{13}	3734.37	
3750.14	4	H_{12}	3750.15	
3753.98	2–3	O III	3754.67	7
		N III	3754.62	6
3759.97	2–3	O III	3759.87	9
3770.72	4	H_{11}	3770.63	
3773.87	1	O III	3774.00	6
3790.82	1	O III	3791.26	6
3798.09	5	H_{10}	3797.90	
3819.71	2–3	HeI	3819.61	4
3835.54	5	H_9	3835.39	
3868.74	4	[Ne III]	3868.7	
3889.05	7*	H_8	3889.05	
		He I	3888.65	10
3905.26	1	Si I	3905.53	10
3926.0	1	He I	3926.53	1
3933.0	2	Ca II	3933.68	200
3961.58	1–2	O III	3961.59	8
3965.04	2–3	He I	3964.73	4
3968.0	1	[Ne III]	3967.5	
		Ca II	3968.49	150
3970.10	7	H_ϵ	3970.08	
4009.43	2	He I	4009.27	1
4026.25	2	He I	4026.19	5

4097.33	2	N III	4097.31	10
4101.79	10	H_δ	4101.75	
4121.03	1–2	He I	4120.81	3
4143.88	1–2	He I	4143.77	2
4340.49	12	H_γ	4340.48	
4363.17	4	[O III]	4363.2	
4387.82	4	He I	4387.93	3
4471.63	3	He I	4471.48	6
4685.72	6	He II	4685.81	
4861.4	15	H_β	4861.34	
4922.	3	He I	4921.93	4
5007.	1	[O III]	5006.84	
5876.	5	He I	5875.62	10
6563.	20	H_α	6562.82	

* The violet wing is weaker than the red wing.

NOTE: Identifications in square brackets designate forbidden transitions.

would be surrounded by a nebulosity giving rise to strong auroral [O III], fairly strong nebular [Ne III], weak nebular [O II] and very weak nebular [O III].

In HD 167362, which contains a late *WC* nucleus exciting a surrounding nebula, we found[5] that the nuclear lines are sharper than is usually observed in Wolf-Rayet stars.

Because of the presence of the strong red component, we cannot settle the question of the excitation of N III; neither is there any information regarding the continuous spectrum of the exciting nucleus. No line of carbon was found.

From 20 bright lines, the radial velocity was found to be +14 km./sec. From three absorption lines (Cr I 4289.72, Fe I 4299.24 and Fe I 4325.76), the radial velocity of the late-type star was found to be +15 km./sec. The two components have practically the same radial velocity.

RW Hydrae belongs to the same group of binaries as AX Persei, CI Cygni, T. Coronae, Z Andromedae, R Aquarii, etc. The excitation of its nebular part is lower than in AX Persei and CI Cygni and is rather similar to R Aquarii; but the intensity ratio of the auroral and nebular transitions is larger in RW Hydrae than in R Aquarii.

[1] $\alpha(1900) = 13^h\,28^m.8$; $\delta\,(1900) = -24°53'$; HD 117970; BD $-24°\,10977$.

[2] *Ap. J.*, **77**, 44 (1933).

[3] *Ap. J.*, **91**, 546 (1940); also Swings, Elvey and Struve, *Pub. A. S. P.*, (in press).

[4] *Lick Obs. Bull.*, **19**, No. 495 (1939).

[5] *Proc. Nat. Acad. Sci.*, **26**, 458 (1940).

COMPOUND MULTIPLICATIVE DIOPHANTINE SYSTEMS

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1. Simple Multiplicative Systems.—The new material in this note begins in § 2, for which the methods recalled in this section are a necessary preliminary.

The power product $x_{i1}^{a_{i1}} \ldots x_{in_i}^{a_{in_i}}$, in which the x's are distinct indeterminates and the a's are integers >0, will be denoted by $(x, a, n)_i$. Hence $(y, b, m)_j, \ldots, (z, c, t)_k$ are defined. A *simple multiplicative system* is a simultaneous system of equations, finite in number, of the form

$$\begin{aligned} a_1(x, a, n)_1 &= a_2(x, a, n)_2 = \ldots = a_i(x, a, n)_i, \\ b_1(y, b, m)_1 &= b_2(y, b, m)_2 = \ldots = b_j(y, b, m)_j, \qquad (1) \\ &\ldots \qquad\qquad \ldots \qquad\qquad \ldots \\ c_1(z, c, t)_1 &= c_2(z, c, t)_2 = \ldots = c_k(z, c, t)_k, \end{aligned}$$

in which $i \geqslant 2, j \geqslant 2, \ldots, k \geqslant 2$, the coefficients $a_1, \ldots, c_k$ are constant integers $\neq 0$, and two or more power products in any one row may or may not have indeterminates in common, while at least two power products in different rows have at least one indeterminate in common. A set of such indeterminates, each of which occurs in at least two rows, will be called a *juncture* of (1).

Two methods are available for finding all rational integer values of the $x, y, \ldots, z$ satisfying the system. In the first,[1] what were called reciprocal arrays furnish a systematic, non-tentative algorithm for exhibiting the complete solution in terms of the minimum number of parameters, independently ranging over all rational integers, necessary and sufficient to give the complete rational integer solution of the system. In the second,[2] the solution of any simple multiplicative system is reduced to finding the primitive solutions of a certain associated system of linear diophantine equations, a problem familiar in the classical theory of algebraic invariants.

An unsolved problem of considerable interest is the *a priori* determination of the minimum number mentioned above in the general case. This number quickly runs into the billions even for systems involving as few as 10 power products in a total of 10 indeterminates none of which occurs to a degree exceeding 10. Thus for even apparently innocuous systems it is impossible to exhibit the complete solution explicitly. For certain types of

systems the minimum number has been determined[2] as a function of the number of indeterminates and their exponents.

2. Compound Multiplicative Systems.—A power product in which at least one indeterminate occurs only to the first degree will be called *elementary*, and any indeterminate occurring only to the first degree in an elementary product will be called *distinguished*. Let each of the $ij\ldots k$ power products in (1) be elementary, and let $x_1, \ldots, x_i$ be distinguished indeterminates in the respective power products in the first row of (1). If $x_1, \ldots, x_i$ are distinct, we shall say that $x_1, \ldots, x_i$ is a *distinguished set* for the first row, and similarly for all rows of (1). A distinguished set need not be unique.

The concept of a distinguished set for the entire system (1) enables us to extend the methods recalled in § 1 to certain types of simultaneous additive systems. We shall say that the set $\sum$ of all the indeterminates in the distinguished sets for all the rows of (1) is a *distinguished set for the system* (1) provided the conditions (A), (B) are satisfied:

(A) *All the indeterminates in* $\sum$ *are distinct;*

(B) Σ *has no indeterminate in common with any juncture of* (1).

A simple multiplicative system need have no distinguished set; if it has one, that one need not be unique. Provided (1) has two or more distinguished sets, any one of them may be chosen as the point of departure for obtaining the complete solution next described.

If (1) has a distinguished set, we call

$$\begin{aligned} \sum_{p=1}^{i} a_p(x, a, n)_p &= 0, \\ \sum_{q=1}^{j} b_q(y, b, m)_q &= 0, \qquad (2) \\ &\cdots \\ \sum_{r=1}^{k} c_r(z, c, t)_r &= 0, \end{aligned}$$

a compound multiplicative system.

THEOREM. *The problem of finding the complete rational integer solution of a compound multiplicative system is reducible to that of finding the complete rational integer solution of a simple multiplicative system; and the complete solution of the compound system exhibits each of the indeterminates in the system as a polynomial with rational integer coefficients in a certain minimum number of parameters ranging independently over all rational integers.*

Denoting the indeterminates in a distinguished set for (2) by small Greek letters, we may rewrite (2) as either of

$$\begin{gathered}\sum_{p=1}^{i} a_p\alpha_p A_p = 0, \quad \sum_{p=1}^{i} \alpha_p A'_p = 0,\\ \sum_{q=1}^{j} b_q\beta_q B_q = 0, \quad \sum_{q=1}^{j} \beta_q B'_q = 0, \qquad (3)\\ \cdots \qquad \cdots \qquad \cdots\\ \sum_{r=1}^{k} c_r\gamma_r C_r = 0, \quad \sum_{r=1}^{k} \gamma_r C'_r = 0,\end{gathered}$$

in which the $A, B, \ldots, C$ are power products, and $A'_p \equiv a_pA_p, B'_q \equiv b_qB_q, \ldots, C'_r \equiv c_rC_r$. The $\alpha_p, A'_p, \beta_q, B'_q, \ldots, \gamma_r, C'_r$ are considered temporarily as distinct indeterminates. The complete rational integer solution of each equation in the second form of (3) is then obtained by an application of a recent theorem due to Dickson.[3] In each solution, exactly half the total number of indeterminates are expressed as power products in a certain minimum number of integer parameters, each parameter entering only to the first degree, while the remaining indeterminates are expressed as polynomials in these parameters and a certain number of new parameters with rational integer coefficients. (It is readily seen that the minimum number for the first equation is $\frac{1}{2}(i^2 + 3i - 2)$, and similarly for all.) The notation may be chosen so that the $\alpha_p, \beta_q, \ldots, \gamma_r$ in the solutions are expressed as the polynomials while the $A'_p, B'_q, \ldots, C'_r$ are expressed as the power products, so that

$$a_pA_p = P_p, b_qB_q = Q_q, \ldots, c_rC_r = R_r,$$
$$(p = 1, \ldots, i;\ q = 1, \ldots, j;\ \ldots;\ r = 1, \ldots, k),$$

where the $P_p, Q_q, \ldots, R_r$ are power products. This is a simple multiplicative system, and hence it is completely solvable by either method in § 1. The values of the indeterminates are obtained as power products. These values are substituted into the polynomial expressions for $\alpha_p, \beta_q, \ldots, \gamma_r$. The method carries with it the proof that the solution of (3) thus obtained is the complete rational integer solution of (2).

3. Generalization of § 2.—It was shown elsewhere[4] that the method applied to solve a simple multiplicative diophantine system completely in rational integers can be extended immediately to obtain the complete solution of such a system in the integral elements of any ring in which the fundamental theorem of arithmetic holds. The passage from simple to compound systems in the ring of rational integers depends only on the existence of a G. C. D. for rational integers. Hence § 2 can be extended

immediately from the special Euclidean ring of rational integers to any Euclidean ring as defined in algebra.[5]

The simplest example is the ring of Gaussian integers. A system (2) of n equations in Gaussian integers is equivalent to a system of $2n$ equations in rational integers. The complete rational integer solution of the $2n$ equations is obtained from that of the original in Gaussian integers by equating reals and imaginaries. More generally, if a ring of algebraic integers of degree g is Euclidean, a system (2) of n equations in integers of the ring is equivalent to a system of gn equations in rational numbers (which can be made integral by a simple transformation of the equations); and the complete solution of the gn equations is obtained from that of the original n equations in integers of the ring by equating coefficients of like elements in the canonical (basis) representation of integers of the ring.

4. Examples.—Specimens of the simpler types of single equations will suffice to indicate the kind of result obtainable by the method. As noted in § 1, the number of parameters in a complete solution increases so rapidly with the degree and number of indeterminates in the system, that it is not possible to exhibit the actual solutions of any but the simplest systems in a reasonable space.

If $c_1, \ldots, c_n$ denote constant rational integers, the complete rational integer solution of

$$c_1x_1 + \ldots + c_nx_n = 0$$

follows from Dickson's theorem.[3] For $n = 3, 4$ the respective numbers of parameters are 3, 6.

If $Q(x_1, \ldots, x_n)$ is a quadratic form in $x_1, \ldots, x_n$ with rational integer coefficients, the complete rational integer solution of

$$Q(x_1, \ldots, x_n) = Q(y_1, \ldots, y_n)$$

is found from the reduction of this equation to the canonical form

$$a_1X_1^2 + \ldots + a_nX_n^2 = a_1Y_1^2 + \ldots + a_nY_n^2,$$

in which $a_1, \ldots, a_n$ are rational integers, and the X, Y are linear homogeneous functions of the x's and y's, respectively, with rational integer coefficients. The solution of the canonical equation follows in an obvious way from that of

$$\sum_{j=1}^{n} a_ju_jv_j = 0,$$

and the solution of this in turn is given by that of

$$\sum_{j=1}^{n} w_ju_jv_j = 0,$$

in which the w, u, v are indeterminates. For $n = 3$ and $a_1a_2a_3 \neq 0$, the solution of the canonical equation is by polynomials of degree 4 in 9 parameters, the coefficients in the polynomials are power products formed from the simultaneous resolution of the coefficients a_1, a_2, a_3 into products of the respective forms $kf_1f_2f_3f_4f_5f_6f_7f_8$, $kf_3f_5f_7g_1g_2g_3g_4g_5$, $kf_2g_3h_1h_2$, where all the letters denote rational integers.

The complete integer solution x, y, z, u, v, w of

$$xu^2 + yv^2 + zw^2 = 0$$

is by means of polynomials in 19 parameters with integer coefficients; the highest degree of a polynomial in the solution is 20 and the lowest 11.

In obtaining the complete integer solution of

$$xu^3 + yv^3 + zw^3 = 0,$$

the central detail is the complete integer solution of the system

$$abc = d^3,\ abe = f^3,\ ag = h^3,$$

for which 31 parameters are required; the respective degrees of a, b, c, d, e, f, g, h in the parameters are 43, 30, 14, 29, 71, 48, 68, 37.

[1] Bell, E. T., *Amer. Jour. Math.*, **55**, 50–66 (1933).
[2] Ward, M., *Ibid.*, **55**, 67–76 (1933).
[3] Dickson, L. E., *Modern Elementary Theory of Numbers*, Chicago, 194 (1939).
[4] Bell, E. T., *Trans. Amer. Math. Soc.*, **35**, 903–914 (1933).
[5] van der Waerden, B. L., *Moderne Algebra*, Berlin, **1**, § 19 (1937).

ON LAGUERRE SERIES

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In an article to appear in the *Annals of Mathematics*, the author discussed certain convergence properties of the series:

$$g(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) G(\lambda_n s) \qquad (1)$$

where (*a*) a_i was a complex constant, $i = 1, 2, 3, \ldots$

(*b*) $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$

$$\lim_{n \to \infty} \lambda_n = \infty$$

(*c*) $s = \sigma + it$, σ and t real,

and where the function $G(z)$ satisfied certain conditions, in particular $G(z)$ was bounded for $R(z) \geq z_0 > 0$. If $G(z) \equiv 1$, then the series (1) is a Dirichlet series of type λ_n; the author named the series (1) a modified exponential series. As a special example the author dealt with such a modifying factor that (1) was a series of Hankel functions of the first kind of fixed order ν.

The condition above on $G(z)$ is very restrictive and prevents the modifying factor $G(z)$ from being a polynomial in z. Consider now the series (1) when $G(z)$ is replaced by the Laguerre polynomial of νth degree, i.e.,

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s) \tag{2}$$

where $L_\nu(z)$ is the Laguerre polynomial of νth degree (fixed), and thus $\exp\left(-\frac{z}{2}\right) L_\nu(z)$ the Laguerre function, and where conditions (a), (b) and (c) still hold.

In order to prove convergence properties of series (2) we shall establish a few lemmas.

LEMMA 1: There exists a constant $K_1 > 0$ and a real number z_0 such that

$$|L_\nu(z)| \geq K_1 > 0 \text{ for } R(z) \geq z_0.$$

Proof: Since $L_\nu(z)$ is a polynomial of the νth degree with ν roots, K_1 and z_0 (positive) can easily be selected satisfying the above.

LEMMA 2: For each $\sigma_0 > 0$, there exists a constant K_2 and an integer N_1 such that whenever $x \geq \lambda_{N_1}$ and $R(s) \geq \sigma_0 > 0$, we have

$$\left| \exp\left(\frac{x(s_0 - s)}{2}\right) \frac{L_\nu(2xs)}{L_\nu(2xs_0)} \right| < K_2.$$

Proof: Given σ_0, we can choose N_1 such that $2\lambda_{N_1}\sigma_0 \geq z_0$ of Lemma 1, and thus for the range $R(s) \geq \sigma_0$ we can divide by $L_\nu(2xs)$. Because of the exponential factor we have

$$\lim_{x \to \infty} \left| \exp\left(\frac{x(s_0 - s)}{2}\right) \frac{L_\nu(2xs)}{L_\nu(2xs_0)} \right| = 0.$$

Separate the interval (λ_{N_1}, ∞) into (λ_{N_1}, x_0) and (x_0, ∞) such that

$$\left| \exp\left(\frac{x(s_0 - s)}{2}\right) \frac{L_\nu(2xs)}{L_\nu(2xs_0)} \right| < A$$

for $x \geq x_0$. Then in the interval (λ_{N_1}, x_0) the function is continuous and has a maximum value, and thus the existence of a constant K_2 and an integer N_1 satisfying the conclusion of the lemma has been established.

LEMMA 3: For each $\sigma_0 > 0$ there exists a constant K_3 and an integer N_1 such that for $R(s) \geq \sigma_0$ and for $x > \lambda_{N_1}$ we have

$$\left|\frac{\frac{\partial L_\nu(2xs_0)}{\partial x}}{L_\nu(2xs_0)} - \frac{\frac{\partial L_\nu(2xs)}{\partial x}}{L_\nu(2xs)}\right| < K_3 |s - s_0|.$$

Proof: Choosing N_1 as in the preceding lemma, we write the above as the integral of its own derivative, and obtain

$$\left|\frac{\frac{\partial L_\nu(2xs_0)}{\partial x}}{L_\nu(2xs_0)} - \frac{\frac{\partial L_\nu(2xs)}{\partial x}}{L_\nu(2xs)}\right| = \left|\int_s^{s_0} 2x \left[\frac{\frac{\partial^2 L_\nu(2xw)}{\partial w \partial x} L_\nu(2xw) - \frac{\partial L_\nu(2xw)}{\partial x}\frac{\partial L_\nu(2xw)}{\partial w}}{L_\nu^2(2xw)}\right] dw\right|.$$

Now because of the polynomial nature of $L_\nu(z)$, the integrand above is bounded for $R(s) \geq \sigma_0$ and for $x \geq \lambda_{N_1}$, and thus can be made less than a constant K_2 and we obtain, on integration, the desired conclusion.

LEMMA 4: For $s_0 = \sigma_0 + it_0$, $\sigma_0 > 0$, there exists a constant N_1 such that for $R(s) \geq \sigma_0$ and $x \geq \lambda_{N_1}$ we have

$$\left|\exp[\lambda_n(s_0 - s)]\frac{L_\nu(2\lambda_n s)}{L_\nu(2\lambda_n s_0)} - \exp[\lambda_{n+1}(s_0 - s)]\frac{L_\nu(2\lambda_{n+1}s)}{L_\nu(2\lambda_{n+1}s_0)}\right| \leq$$

$$K_2(1 + K_3)\frac{2|s - s_0|}{R(s - s_0)}\left\{\exp\left[\frac{\lambda_n R(s_0 - s)}{2}\right] - \exp\left[\frac{\lambda_{n+1}R(s_0 - s)}{2}\right]\right\}.$$

Proof: Writing the function as the integral of its own derivative we have

$$\left|\exp[\lambda_n(s_0 - s)]\frac{L_\nu(2\lambda_n s)}{L_\nu(2\lambda_n s_0)} - \exp[\lambda_{n+1}(s_0 - s)]\frac{L_\nu(2\lambda_{n+1}s)}{L_\nu(2\lambda_{n+1}s_0)}\right| =$$

$$\left|\int_{\lambda_n}^{\lambda_{n+1}}\left\{(s_0 - s)\exp[x(s_0 - s)]\frac{L_\nu(2xs)}{L_\nu(2xs_0)} + \right.\right.$$

$$\left.\left.\exp[x(s_0 - s)]\frac{L_\nu(2xs)}{L_\nu(2xs_0)}\left[\frac{\frac{\partial L_\nu(2xs)}{\partial x}}{L_\nu(2xs)} - \frac{\frac{\partial L_\nu(2xs_0)}{\partial x}}{L_\nu(2xs_0)}\right]\right\} dx\right| \leq$$

$$\int_{\lambda_n}^{\lambda_{n+1}}\left|(s_0 - s)K_2\exp\left[\frac{x(s_0 - s)}{2}\right] + \exp\left[\frac{x(s_0 - s)}{2}\right]K_2 \cdot K_3(s_0 - s)\right| dx \leq$$

$$K_2(1 + K_3)|s_0 - s|\int_{\lambda_n}^{\lambda_{n+1}}\left|\exp\left[\frac{x(s_0 - s)}{2}\right]\right| dx \leq$$

$$\frac{K_2(1 + K_3)2|s_0 - s|}{R(s - s_0)}\left\{\exp\left[\frac{\lambda_n R(s_0 - s)}{2}\right] - \exp\left[\frac{\lambda_{n+1}R(s_0 - s)}{2}\right]\right\}.$$

LEMMA 5: *Abel's Lemma on Partial Summation.*—We have identically:

$$\sum_{n=p}^{n=q} U_n V_n \equiv \sum_{n=p}^{n=q-1} \left[(V_n - V_{n+1}) \sum_{m=p}^{m=n} U_m \right] + V_q \sum_{m=p}^{m=q} U_m.$$

THEOREM 1: If $f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s)$ is convergent for $s = s_0 = \sigma_0 + it_0$ where $\sigma_0 > 0$, then it is uniformly convergent throughout the angular sector defined by $|am(s - s_0)| \leq \psi < \frac{\pi}{2}$.

Proof: Since $\sigma_0 > 0$, we can select the integer N_1 of Lemmas 2, 3 and 4. In the infinite angular sector $|am(s - s_0)| \leq \psi < \frac{\pi}{2}$, the finite summation $\sum_{n=1}^{k} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s)$ gives no trouble, since each term is bounded.

Thus we need only show that given $\epsilon_1 > 0$, we can find an integer n_1 such that for $q \geq p \geq n_1$ we have

$$\left| \sum_{n=p}^{n=q} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s) \right| < \epsilon_1 \tag{3}$$

for all s in the angular sector.

Applying Abel's lemma we get

$$\left| \sum_{n=p}^{n=q} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s) \right| \leq$$

$$\sum_{n=p}^{n=q-1} \left| \left\{ \exp[\lambda_n(s_0 - s)] \frac{L_\nu(2\lambda_n s)}{L_\nu(2\lambda_n s_0)} - \exp[\lambda_{n+1}(s_0 - s)] \frac{L_\nu(2\lambda_{n+1} s)}{L_\nu(2\lambda_{n+1} s_0)} \right\} \right| \cdot \tag{4}$$

$$\cdot \left| \sum_{m=p}^{m=n} a_m \exp[-\lambda_m s_0] L_\nu(2\lambda_m s_0) \right| +$$

$$\left| \exp[\lambda_q(s_0 - s)] \frac{L_\nu(2\lambda_q s)}{L_\nu(2\lambda_q s_0)} \right| \cdot \left| \sum_{m=p}^{m=q} a_m \exp(-\lambda_m s_0) L_\nu(2\lambda_m s_0) \right| .$$

Now given σ_0, ϵ_1 and ψ (each a constant) we can select $\epsilon = \epsilon(\sigma_0, \epsilon_1, \psi)$; the exact functional form will be specified later. Given this ϵ, we can select N_2 such that

$$\left| \sum_{m=p}^{m=q} a_m \exp[-\lambda_m s_0] L_\nu(2\lambda_m s_0) \right| < \epsilon \tag{5}$$

for $q \geq p \geq N_2$.

Substituting (5) into (4) and using the result of Lemma 4 we obtain, where $n \geq n_1 = \max[N_1, N_2]$,

$$\left|\sum_{n=p}^{n=q} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s)\right| \leq$$

$$\epsilon K_2(1+K_3)\frac{2|s-s_0|}{R(s-s_0)}\sum_{n=p}^{n=q-1}\left\{\exp\left[\frac{\lambda_n R(s_0-s)}{2}\right]-\exp\left[\frac{\lambda_{n+1}R(s_0-s)}{2}\right]\right\} \tag{6}$$

$$+\epsilon K_2 \exp\left[\frac{\lambda_q R(s_0-s)}{2}\right] \leq$$

$$\epsilon K_2(1+K_3)\frac{2|s-s_0|}{R(s-s_0)}\exp\left[\frac{\lambda_q R(s_0-s)}{2}\right] \leq$$

$$\epsilon K_2(1+K_3)\cdot 2\cdot \sec\psi$$

since $\dfrac{|s-s_0|}{R(s-s_0)} = \sec\{|a_m(s-s_0)|\} \leq \sec\psi$.

Thus by choosing $\epsilon = \dfrac{\epsilon_1}{2K_2(\sigma_0)[1+K_3(\sigma_0)]\sec\psi}$ we see that (3) is satisfied and the uniform convergence of the Laguerre series in the angular sector is established.

THEOREM 2: If the Laguerre series is convergent for $s = s_0 = \sigma_0 + it_0$ where $\sigma_0 > 0$, then it is convergent in the half plane $R(s) > \sigma_0$.

Proof: This theorem is implicit in the more general theorem 1.

We can thus define an *abscissa of convergence* α and a *line of convergence* $R(s) = \alpha$ for the Laguerre series (2) by taking a Dedekind cut of the non-negative real numbers. Theorem 3, which follows, shows that we can define an *abscissa of absolute convergence* β and a *line of absolute convergence* $R(s) = \beta$. Notice, as is also the case with Dirichlet series and the modified exponential series (1), that convergence is proved only for $R(s) > \alpha$ while absolute convergence will be proved for $R(s) \geq \beta$.

THEOREM 3: If $f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) L_\nu(2\lambda_n s)$ is absolutely convergent for $s = s_1 = \sigma_1 + it_1$, where $\sigma_1 > 0$, then it is absolutely convergent in the half plane $R(s) \geq \sigma_1$.

Proof: We shall show that for $R(s) \geq \sigma_1$, given $\epsilon_2 > 0$, we can find an integer n_2 such that for $q \geq p \geq n_2$ we have

$$\sum_{n=p}^{n=q} |a_n| \exp(-\lambda_n \sigma)|L_\nu(2\lambda_n s)| < \epsilon_2.$$

By hypothesis, given $\epsilon > 0$, there exists an integer N_3 such that for $q \geq p \geq N_3$ we have

$$\sum_{n=p}^{n=q} |a_n| \exp(-\lambda_n \sigma_1)|L_\nu(2\lambda_n s_1)| < \epsilon.$$

Take $n_2 = \max [N_3, N_1]$ where N_1 is given by Lemma 2. We have

$$\sum_{n=p}^{n=q} |a_n| \exp(-\lambda_n \sigma) |L_\nu(2\lambda_n s)| \leq$$

$$\sum_{n=p}^{n=q} |a_n| \exp(-\lambda_n \sigma_1) |L_\nu(2\lambda_n s_1)| \left| \frac{L_\nu(2\lambda_n s)}{L_\nu(2\lambda_n s_1)} \right| \exp[(\sigma_1 - \sigma)\lambda_n] \leq \epsilon K_2$$

where K_2 is given by Lemma 2, and thus by proper choice of ϵ the desired inequality is obtained.

EQUILONG AND CONFORMAL TRANSFORMATIONS OF PERIOD TWO

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1. Introduction.—Our main problem is to find all equilong transformations of period two. We find that in equilong geometry the set of all equilong transformations of period two may be classified into *three* distinct types: (T_1) equilong involutions, (T_2) K symmetries and (T_3) D inversions. This is in contrast with the conformal theory where Kasner has proved that the set of all conformal transformations of period two consists of *two* distinct types: ($\mathfrak{T}_1$) conformal involutions, and ($\mathfrak{T}_2$) conformal symmetries (the Schwarzian reflections).

We thus have five distinct types of transformations of period two: three equilong and two conformal.

The conformal types (and the groups generated by them) were discussed by Kasner.[1] In this and later papers, we are going to discuss the analogous situations in equilong geometry.

We shall consider only those equilong transformations of the plane which convert the positive y_1-axis (with hessian or equilong coördinates (0, 0)) into itself and are regular at the positive y_1-axis. Such regular equilong transformations are expressed by power series of the two forms

$$Z = \alpha_1 z + \alpha_2 z^2 + \ldots, \quad \alpha_1 \alpha_1 \neq 0, \tag{1.1}$$

$$\overline{Z} = \alpha_1' z + \alpha_2' z^2 + \ldots, \quad \alpha_1' \bar{\alpha}_1' \neq 0, \tag{1.2}$$

where $\bar{z} = x - jy$ is the conjugate of the dual variable $z = x + jy$ with $j^2 = 0$ and (x, y) the hessian or equilong coördinates of a line, and the coefficients are arbitrary dual numbers of the form $a + jb$. Separating these equations into the two real forms, we find

$$X = a_1x + a_2x^2 + \ldots,$$
$$Y = y(a_1 + 2a_2x + \ldots) + (b_1x + b_2x^2 + \ldots), \qquad (2.1)$$

$$X = a_1'x + a_2'x^2 + \ldots,$$
$$Y = -y(a_1' + 2a_2'x + \ldots) - (b_1'x + b_2'x^2 + \ldots), \qquad (2.2)$$

The *direct* equilong transformations (1.1) or (2.1) form a continuous infinite group G. The *reverse* equilong transformations (1.2) or (2.2) do not form a group. However, if we add this set to our group G, we obtain a mixed group G'.

We determine all regular equilong transformations of period two. In the direct type $Z = f(z)$ the functional equation is $f[f(z)] \equiv z$, that is, $f^2 = 1$; and in the reverse type $\bar{Z} = f(z)$ the functional equation is $\bar{f}[f(z)] \equiv z$, that is, $\bar{f}f = 1$ where $\bar{f}$ denotes the series whose coefficients are the conjugates of those of the series f. Thus we are to solve two types of functional equations. The first type $f^2 = 1$ yields as solutions *a single* set (T_1) of equilong involutions (besides the identity). On the other hand, the second type $\bar{f}f = 1$ admits as solutions *two equilongly distinct* sets (T_2) of K symmetries and (T_3) of D inversions.

To find the regular conformal transformations of period two, the analogous functional equations $f^2 = 1$ and $\bar{f}f = 1$ have to be solved. The first type $f^2 = 1$ yields as solutions *a single* set $(\mathfrak{T}_1)$ of conformal involutions (besides the identity). The second type $\bar{f}f = 1$ admits as solutions *a single* set $(\mathfrak{T}_2)$ of conformal symmetries (Schwarzian reflections).

It can be shown that (T_1) any equilong involution can be reduced equilongly to the simple form $Z = -z$ (symmetry in the positive y_1-axis), (T_2) any K symmetry can be changed equilongly to the simple form $Z = \bar{z}$ (symmetry in the origin accompanied by reversal of orientation) and (T_3) any D inversion can be reduced equilongly to the simple form $Z = -\bar{z}$ (symmetry in the x_1-axis accompanied by reversal of orientation). These simple symmetries are to be regarded as transformations between the lines of the plane.

In the conformal theory, we recall that $(\mathfrak{T}_1)$ any conformal involution can be changed conformally to the simple form $Z_1 = -z_1$ (symmetry in the origin), and $(\mathfrak{T}_2)$ any conformal symmetry may be reduced conformally to the simple form $Z_1 = \bar{z}_1$ (symmetry in the x_1-axis), where $\bar{z}_1 = x_1 - iy_1$ is the conjugate of the complex variable $z_1 = x_1 + iy_1$ with $i^2 = -1$ and (x_1, y_1) the cartesian coördinates of a point. These symmetries are considered to be transformations between the points of the plane.

2. The Discussion of (T_1) Equilong Involutions.—To obtain the equations of any equilong involution in the implicit form, it is convenient to rewrite any direct equilong transformation (1.1) in another form. Let $\lambda = Z + z$, $\mu = Z - z$ so that $Z = {}^1/_2(\lambda + \mu)$, $z = {}^1/_2(\lambda - \mu)$. Substituting these results into (1.1), we find

$$\tfrac{1}{2}(1 - \alpha_1)\lambda + \tfrac{1}{2}(1 + \alpha_1)\mu = \alpha_2/2^2(\lambda - \mu)^2 + \ldots. \qquad (3)$$

If $\alpha_1 \neq -1 + jb$ (b real), we can solve this for μ as a power series in λ. On the other hand, if $\alpha_1 \neq 1 + jb$ (b real), we can solve this for λ as a power series in μ. Hence in all cases, any direct equilong transformation can be written in either or both of the two forms

$$Z - z = \beta_1(Z + z) + \beta_2(Z + z)^2 + \ldots, \qquad (4.1)$$

$$Z + z = \gamma_1(Z - z) + \gamma_2(Z - z)^2 + \ldots. \qquad (4.2)$$

Thus the group G of all direct equilong transformations has been subdivided into two sets (4.1) and (4.2) which are *not* mutually exclusive.

Since any direct equilong transformation, defined implicitly by $F(Z, z) = 0$, is an involution if and only if it is symmetric in the two variables, that is, $F(Z, z) \equiv F(z, Z)$, we obtain from (4.1) and (4.2) the following result.

THEOREM 1. *Any equilong involution is given in the implicit form by*

$$Z + z = \alpha_2(Z - z)^2 + \alpha_4(Z - z)^4 + \ldots, \qquad (5)$$

where the α_{2n} are arbitrary dual numbers. In the real implicit form, any equilong involution is

$$X + x = a_2(X - x)^2 + a_4(X - x)^4 + \ldots,$$

$$Y + y = (Y - y)[2a_2(X - x) + 4a_4(X - x)^3 + \ldots] + [b_2(X - x)^2 + b_4(X - x)^4 + \ldots]. \qquad (6)$$

Any equilong involution in the explicit dual variable form is

$$Z = -z + \gamma_2 z^2 - \gamma_2{}^2 z^3 + \gamma_4 z^4 + (-3\gamma_2\gamma_4 + 2\gamma_2{}^4)z^5 + \ldots, \qquad (7)$$

where the γ_{2n} are arbitary dual numbers. Finally in the real explicit form any equilong involution is

$$X = -x + c_2x^2 - c_2{}^2x^3 + c_4x^4 + (-3c_2c_4 + 2c_2{}^4)x^5 + \ldots,$$

$$Y = y[-1 + 2c_2x - 3c_2{}^2x^2 + 4c_4x^3 + 5(-3c_2c_4 + 2c_2{}^4)x^4 + \ldots] + [d_2x^2 - 2c_2d_2x^3 + d_4x^4 + (-3c_2d_4 - 3c_4d_2 + 8c_2{}^3d_2)x^5 + \ldots]. \qquad (8)$$

3. The Discussion of (T_2) K Symmetries.[2]—To find the equations of any K symmetry and of any D inversion, it is convenient to rewrite any reverse equilong transformation (1.2) in another form. By an argument very similar to the one given at the beginning of section 2, we find in all cases that any reverse equilong transformation can be written in either or both of the two forms

$$\bar{Z} - z = \beta_1(\bar{Z} + z) + \beta_2(\bar{Z} + z)^2 + \ldots, \qquad (9.1)$$

$$\bar{Z} + z = \gamma_1(\bar{Z} - z) + \gamma_2(\bar{Z} - z)^2 + \ldots. \qquad (9.2)$$

The set of all reverse equilong transformations has been subdivided into two sets (9.1) and (9.2) which are *not* mutually exclusive.

A *K symmetry* is any reverse equilong transformation of period two which is of the form (9.1). On the other hand, any *D inversion* is any reverse equilong transformation of period two which is of the form (9.2).

Since the inverse of any K symmetry is itself, the solution for $\bar{Z}$ of the equation $\bar{Z} - z = f(\bar{Z} + z)$ must be identical with the solution for z of the conjugate equation $\bar{z} - Z = -\bar{f}(\bar{z} + Z)$. Therefore $\beta_n = -\bar{\beta}_n$ for all n. Thus the power series expansion in $\bar{Z} + z$ has as coefficients only *pure* dual numbers. Hence

THEOREM 2. *Any K symmetry is given in the implicit form by the equation*

$$\bar{Z} - z = j[b_1(\bar{Z} + z) + b_2(\bar{Z} + z)^2 + \ldots], \tag{10}$$

where the b_n are real numbers. In the real explicit form, any K symmetry is

$$X = x, \; Y = -y + 2(d_1x + d_2x^2 + \ldots). \tag{11}$$

In the explicit dual variable form, any K symmetry may be written as

$$\bar{Z} = z - 2j(d_1z + d_2z^2 + \ldots). \tag{12}$$

4. The Discussion of (T_3) D Inversions.—Observing that the inverse of any D inversion is itself, the solution for $\bar{Z}$ of the equation $\bar{Z} + z = f(\bar{Z} - z)$ must be identical with the solution for $\bar{z}$ of the conjugate equation $\bar{z} + Z = \bar{f}[-(\bar{z} - Z)]$. Therefore $\gamma_{2n-1} = -\bar{\gamma}_{2n-1}$ and $\gamma_{2n} = \bar{\gamma}_{2n}$ for all n. Thus in the power series expansion in $(\bar{Z} - z)$, the odd coefficients must be *pure* dual numbers and the even coefficients must be real numbers. Hence

THEOREM 3. *Any D inversion is given in the implicit form by the equation*

$$\bar{Z} + z = [a_2(\bar{Z} - z)^2 + a_4(\bar{Z} - z)^4 + \ldots] + j[b_1(\bar{Z} - z) + b_3(\bar{Z} - z)^3 + \ldots], \tag{13}$$

where the a_{2n} and the b_{2n-1} are real numbers. In the real implicit form, any D inversion may be written as

$$X + x = a_2(X - x)^2 + a_4(X - x)^4 + \ldots,$$

$$Y - y = (Y + y)[2a_2(X - x) + 4a_4(X - x)^3 + \ldots] - [b_1(X - x) + b_3(X - x)^3 + \ldots]. \tag{14}$$

Finally in the real explicit form any D inversion is

$$X = -x + c_2x^2 - c_2^2x^3 + c_4x^4 + (-3c_2c_4 + 2c_2^4)x^5 + \ldots,$$

$$Y = -y[-1 + 2c_2x - 3c_2^2x^2 + 4c_4x^3 + 5(-3c_2c_4 + 2c_2^4)x^4 + \ldots] -$$

$$[d_1x - {}^3/_2c_2d_1x^2 + d_3x^3 + {}^5/_2(-c_2d_3 - c_4d_1 + 2c_2^3d_1)x^4 + d_5x^5 + \ldots]. \tag{15}$$

5. *The Reduction of the* (T_1) *Equilong Involutions,* (T_2) *K Symmetries and* (T_3) *D Inversions to Canonical Forms.*—In the first place, it may be shown that there exists formally at least one real function $\Phi(x) = \sum c_n x^n$ $(c_1 \neq 0)$ such that $\Phi(X) + \Phi(x) \equiv 0$ where X is any real function of period two, given implicitly by the first of equations (6) or (14). It results that all the even coefficients c_{2n} are determined as linear functions of the odd coefficients c_{2n-1} such that the coefficients of these linear functions are polynomials in the a_{2n}. Thus there exist many such functions $\Phi(x)$.

Under the direct equilong transformation $X = \Phi(x)$, $Y = y\Phi_x(x)$, our equilong involution (6) becomes $X = -x$, $Y = -y + 2(d_2x^2 + d_4x^4 + \ldots)$. The transform of this under the direct equilong transformation $X = x$, $Y = y - (d_2x^2 + d_4x^4 + \ldots)$ is the equilong involution $X = -x$, $Y = -y$.

Under the direct equilong transformation $X = x$, $Y = y - (d_1x + d_2x^2 + \ldots)$, the K symmetry (11) is reduced to the K symmetry $X = x$, $Y = -y$.

Under the direct equilong transformation $X = \Phi(x)$, $Y = y\Phi_x(x)$, our D inversion (14) becomes $X = -x$, $Y = y + 2(d_1x + d_3x^3 + \ldots)$. The transform of this under the direct equilong transformation $X = x$, $Y = y + (d_1x + d_3x^3 + \ldots)$ is the D inversion $X = -x$, $Y = y$. Therefore, we discover the following result.

THEOREM 4. *Any equilong involution can be reduced equilongly to the symmetry through the positive* y_1*-axis* $Z = -z$. *Any K symmetry may be changed equilongly to the symmetry through the origin accompanied by reversal of orientation* $Z = z$. *Any D inversion may be reduced equilongly to the symmetry through the* x_1*-axis accompanied by reversal of orientation* $Z = -z$.

It may be shown without any difficulty that there do *not* exist any direct or reverse equilong transformations which will carry any one of the three canonical forms (T_1) $X = -x$, $Y = -y$, (T_2) $X = x$, $Y = -y$, (T_3) $X = -x$, $Y = y$ into the other. From this, we may derive the following proposition.

THEOREM 5. *All the equilong transformations of period two consist of the three equilongly mutually exclusive sets:* (T_1) *equilong involutions,* (T_2) *K symmetries and* (T_3) *D inversions.*

We note that D inversion is equilongly equivalent to Laguerre inversion, whereas K symmetry is equivalent to K inversion (with respect to a circle). Any K symmetry is the product of three D inversions. But no D inversion can be factored into any number of K symmetries.

In the conformal theory, any conformal symmetry is conformally equivalent to Moebius inversion. Thus we have *three* distinct types of inversion in circle geometry: (C_1) Moebius inversion, (C_2) Laguerre inversion and (C_3) K inversion.

[1] Kasner, "Infinite Groups Generated by Conformal Transformations of Period Two (Involution and Symmetries)," *Am. Jour. Math.*, **38**, 177–184 (1916); Kasner, "Conformal Geometry," *Proc. Int. Cong. Math.*, **2**, 81 (1912).

[2] Kasner, "Equilong Symmetry with Respect to Any Curve," *Proc. Nat. Acad. Sci.*, **26**, 287–291 (1940).

[3] Kasner, "Geometry of Conformal Symmetry (Schwarzian Reflection)," *Ann. Math.*, **38**, 873–879 (1937); Pfeiffer, "On the Conformal Geometry of Analytic Arcs," *Am. Jour. Math.*, **37**, 395–430 (1915); Comenetz, "Conformal Geometry on a Surface," *Ann. Math.*, **39**, 863–871 (1938).